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CENTRE FOR DISTANCE AND ONLINE EDUCATION

(CDOE)

BACHELOR OF COMPUTER SCIENCE

SEMESTER - II

ELECTIVE COURSE: NUMERICAL METHODS - I

(Candidates admitted from 2024 onwards)

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B. Sc., COMPUTER SCIENCE

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ELECTIVE COURSE: NUMERICAL METHODS - I

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Unit - I

Solutions of Algebraic and Transcendental equations

1.0 Introduction:

The limitations of analytical methods for the solution of equations have necessitated the use of iterative methods. An iterative method begins with an approximate value of the root which is generally obtained with the help of Intermediate value property of the equation. This initial approximation is then successively improved iteration by iteration and this process stops when the desired level of accuracy is achieved. The following methods are used to find the roots of the given equation.

1. The Bisection Method
2. The iteration Method
3. The method of False Position
4. The Newton – Raphson Method.

1.1 Properties of equation:

(i) If f(a) and f(b) have opposite signs then one root of f(x) = 0 lies between a and b.

(ii) Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term.

(iii) Every equation of an even degree with last term negative has at least a pair of real roots one positive and other negative.

1.2 Bisection Method or Interval Halving Method or BOLZANO's Method

Suppose we have an equation of the form $f(x) = 0$ whose solution in the range (a,b). We also assume that $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are of opposite signs, atleast one real root between a and b should exist. We assume that root to be $x_0 = \frac{a+b}{2}$. Now find the sign of $f(x_0)$. If $f(x_0)$ is negative then the root lies between a and x_0. If $f(x_0)$ is positive then the root lies between x_0 and b. In this way taking the midpoint of the range as the approximate root, we form a sequence of approximate roots $x_0, x_1, x_2, \ldots$ whose limit of convergence is the exact root.

Example 1. Find the positive root of $x^3 - x = 1$ correct to four decimal places by bisection method.

Solution:

Let $f(x) = x^3 - x - 1$

Here $f(0) = -1 = -ve, f(1) = -ve$, $f(2) = 5 = +ve$

Hence the root lies between 1 and 2. Let $I = [1,2]$

Let $x_0 = \frac{1+2}{2} = 1.5$

Now $f(x_0) = f(1.5) = +ve$ and $f(1) = -ve$

Hence the root lies between 1 and 1.5.

Let $x_1 = \frac{1+1.5}{2} = 1.25$

Now $f(x_1) = f(1.25) = -ve$ and $f(1.5) = +ve$

Hence the root lies between 1.25 and 1.5.

Let $x_2 = \frac{1.25+1.5}{2} = 1.375$

Now $f(x_2) = f(1.375) = +ve$

Hence the root lies between 1.25 and 1.375.

Let $x_3 = \frac{1.25+1.375}{2} = 1.3125$

Now $f(x_3) = f(1.3125) = -ve$

Hence the root lies between 1.3125 and 1.375.

Let $x_4 = \frac{1.3125+1.375}{2} = 1.3438$

Now $f(x_4) = f(1.3438) = +ve$

Hence the root lies between 1.3125 and 1.3438.

Let $x_5 = \frac{1.3125+1.3438}{2} = 1.3282$

Now $f(x_5) = f(1.3282) = +ve$

Hence the root lies between 1.3125 and 1.3282.

Let $x_6 = \frac{1.3125+1.3282}{2} = 1.3204$

Now $f(x_6) = f(1.3204) = -ve$

Hence the root lies between 1.3204 and 1.3282.

Let $x_7 = \frac{1.3204+1.3282}{2} = 1.3243$

Now $f(x_7) = f(1.3243) = -ve$

Hence the root lies between 1.3243 and 1.3282.

Let $x_8 = \frac{1.3243+1.3282}{2} = 1.3263$

Now $f(x_8) = f(1.3263) = +ve$

Hence the root lies between 1.3243 and 1.3263.

Let $x_9 = \frac{1.3243+1.3263}{2} = 1.3253$

Now $f(x_9) = f(1.3253) = +ve$

Hence the root lies between 1.3243 and 1.3253.

Let $x_{10} = \frac{1.3243+1.3253}{2} = 1.3248$

Now $f(x_{10}) = f(1.3248) = +ve$

Hence the root lies between 1.3243 and 1.3248.

Let $x_{11} = \frac{1.3243+1.3248}{2} = 1.3246$

Now $f(x_{11}) = f(1.3246) = -ve$

Hence the root lies between 1.3248 and 1.3246.

Let $x_{12} = \frac{1.3248+1.3246}{2} = 1.3246$

Therefore, the approximate root is 1.3246.

Example 2. Find the root of $x - \cos x = 0$ by bisection method.

Solution:

Let $f(x) = x - \cos x$

$f(0) = -ve, f(0.5) = -ve, f(1) = +ve$

Hence the root lies between 0.5 and 1.

Let $x_0 = \frac{0.5+1}{2} = 0.75$

Now $f(x_0) = f(0.75) = +ve$

Hence the root lies between 0.5 and 0.75.

Let $x_1 = \frac{0.5+0.75}{2} = 0.625$

Now $f(x_1) = f(0.625) = -ve$

Hence the root lies between 0.625 and 0.75.

Let $x_2 = \frac{0.625+0.75}{2} = 0.6875$

Now $f(x_2) = f(0.6875) = -ve$

Hence the root lies between 0.6875 and 0.75.

Let $x_3 = \frac{0.6875+0.75}{2} = 0.7188$

Now $f(x_3) = f(0.7188) = -ve$

Hence the root lies between 0.7188 and 0.75.

Let $x_4 = \frac{0.7188+0.75}{2} = 0.7344$

Now $f(x_4) = f(0.7344) = -ve$

Hence the root lies between 0.7344 and 0.75.

Let $x_5 = \frac{0.7344+0.75}{2} = 0.7422$

Now $f(x_5) = f(0.7422) = +ve$

Hence the root lies between 0.7344 and 0.7422.

Let $x_6 = \frac{0.7344+0.7422}{2} = 0.7383$

Now $f(x_6) = f(0.7383) = -ve$

Hence the root lies between 0.7383 and 0.7422.

Let $x_7 = \frac{0.7383+0.7422}{2} = 0.7402$

Now $f(x_7) = f(0.7402) = +ve$

Hence the root lies between 0.7383 and 0.7402.

Let $x_8 = \frac{0.7383+0.7402}{2} = 0.7393$

Now $f(x_8) = f(0.7393) = +ve$

Hence the root lies between 0.7383 and 0.7393.

Let $x_9 = \frac{0.7383+0.7393}{2} = 0.739$

Therefore, the approximate root is 0.739. (Correct to three decimal places)

1.3 Fixed-point Iteration Method

In open methods, we have to use a formula to predict the root, as an example for the fixed-point iteration method we can rearrange f(x) so that x is on the left-hand side of the equation:

$$x = \emptyset(x)$$

This can be achieved by algebraic manipulation of by simply adding x to both sides of the original equation, example:

$$x^2 + 3x - 1 = 0 \Leftrightarrow x = \frac{1-x^2}{3}$$

$$\sin(x) = 0 \Leftrightarrow x = \sin(x) + x$$

This is important since it will allow us to develop a formula to predict the new value for x as a function of an old value of x as

$$x_{i+1} = \emptyset(x_i).$$

The order of convergence of this method is linear.

Example 1: Use the iteration method to find a root of the equation x = ½ + sin x?

Solution:

Let f(x) = sin x – x + ½

f(1) = sin 1 – 1 + ½ = 0.84 – 0.5 = +ve

f(2) = sin 2 – 2 + ½ = 0.9.9 – 1.5 = -ve.

A root lies between 1 and 2. The given equation can be written as

x = sin x + ½ = $\emptyset(x)$

$|\emptyset'(x)| = |\cos x| < 1 \; in \; [1,2]$

Hence the iteration method can be applied. Let the approximation be $x_0 = 1$.

The successive approximation is as follows:

$x_1 = \emptyset(x_0)$ = sin 1 + ½ = 0.8414 + 0.5 = 1.3414

$x_2 = \emptyset(x_1)$ = sin (1.3414) + ½ = 0.9738 + 0.5 = 1.4738

$x_3 = \emptyset(x_2)$ = sin (1.4738) + ½ = 0.9952 + 0.5 = 1.4952

$x_4 = \emptyset(x_3)$ = sin (1.4952) + ½ = 0.9971 + 0.5 = 1.4971

$x_5 = \emptyset(x_4)$ = sin (1.4971) + ½ = 0.9972 + 0.5 = 1.4972

Since x_4 and x_5 are almost equal and the required root is 1.497.

Example 2: Find the positive root of $3x - \sqrt{1+\sin x} = 0$ by iteration method.

Solution:

Let $f(x) = 3x - \sqrt{1+\sin x}$

$$f(0) = -ve \; and \; f(1) = +ve$$

The root lies between 0 and 1

The given equation can be written as $x = \frac{1}{3}\sqrt{1+\sin x} = \emptyset(x), \emptyset'(x) = \frac{\cos x}{6\sqrt{1+\sin x}}$

In (0,1), $|\emptyset'(x)| < 1$ for all x, so we can use iteration method.

Take $x_0 = 0.4$,

$$x_1 = \frac{1}{3}\sqrt{1+\sin(0.4)} = 0.3929$$

$$x_2 = \frac{1}{3}\sqrt{1+\sin(0.3929)} = 0.3919$$

$$x_3 = \frac{1}{3}\sqrt{1+\sin(0.3919)} = 0.3918$$

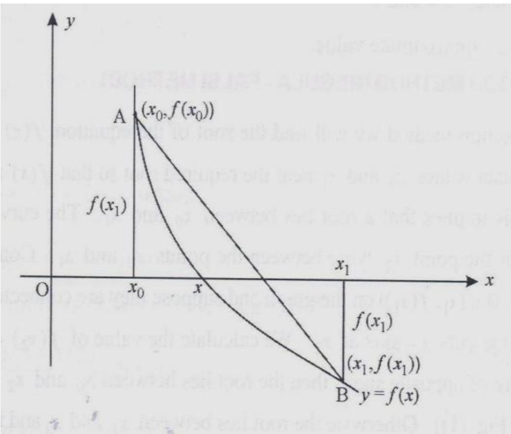
$$x_4 = \frac{1}{3}\sqrt{1+\sin(0.3918)} = 0.3919$$

$$x_4 = \frac{1}{3}\sqrt{1+\sin(0.3919)} = 0.3919$$

Therefore the root is 0.3919.

1.4 Regula Falsi method or the method of False position

In the false position method we will find the root of the equation $f(x) = 0$, Consider two initial approximate values x_0 and x_1 near the required root so that $f(x_0)$ and $f(x_1)$ have different signs. This implies that a root lies between x_0 and x_1. The curve $f(x)$ Crosses x-axis only once at the Point x_2 lying between the points x_0 and x_1. Consider the point $A = (x_0, f(x_0))$ and $B = (x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis at x_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2 and value x_1 is replaced by x_2 otherwise the root lies between x_2 and x_1 and



the value of x_0 is replaced by x_2. Another line is drawn by connecting the newly obtained pair of values. Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points $x_2, x_3, x_4, \ldots$ obtained converge to the expected root of the equation $f(x) = 0$.

If $x_0 = a$ and $x_1 = b$ then $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$. The order of convergence of this method is 1.618.

Example 1: Solve for a positive root of $x^3 - 4x + 1 = 0$ by Regula Falsi method.

Solution:

Let $f(x) = x^3 - 4x + 1$

$f(1) = -2 = -ve$, $f(0) = 1 = +ve$

Therefore a root lies between 0 and 1.

Let a = 0 and b = 1

$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = 0.3333$

$f(x_1) = f(0.3333) = -0.2963$ = -ve

Therefore the root lies between 0 and 0.3333

Therefore $x_2 = \frac{0(0.3333) - 0.3333(1)}{0.3333 - 1} = 0.2571$

Now $f(x_2) = f(0.2571) = -0.0116$ = -ve

Therefore the root lies between 0 and 0.2571

$x_3 = \frac{0*f(0.2571) - 0.2574*f(0)}{f(0.2571) - f(0)} = 0.2542$

Now $f(x_3) = f(0.2542) = -0.0004$

Therefore the root lies between 0 and 0.2542

$x_4 = \frac{0*f(0.2542) - 0.2542*f(0)}{f(0.2542) - f(0)} = 0.2541$

Now $f(x_4) = f(0.2541) = -0.00001$

Therefore the root lies between 0 and 0.2541

$x_5 = \frac{0*f(0.2541) - 0.2541*f(0)}{f(0.2541) - f(0)} = 0.2541$

Hence the root is 0.2541.

Example 2: Find an approximate root of $x \log_{10} x - 1.2 = 0$ by method of False position.

Solution:

Let $f(x) = x \log_{10} x - 1.2$

$f(2) = -0.5979 = -ve \ \ and \ f(3) = 0.2314 = +ve$

Hence the root lies between 2 and 3. And let a=2 and b=3

$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = 2.7210$

$f(x_1) = f(2.7210) = -0.0171$ = -ve

Hence the root lies between 2.7210 and 3

Therefore $x_2 = \frac{2.7210*f(3)-3*f(2.7210)}{f(3)-f(2.7210)} = 2.7402$

Now $f(x_2) = f(2.7402) = -0.0004$ = -ve

Therefore the root lies between 2.7402 and 3

$x_3 = \frac{2.7402*f(3)-3*f(2.7402)}{f(3)-f(2.7402)} = 2.7406$

Now $f(x_3) = f(2.7406) = 0.00011$

Therefore the root lies between 2.7402 and 2.7406

$x_4 = \frac{2.7402*f(2.7406)-2.7406*f(2.7402)}{f(2.7406)-f(2.7402)} = 2.7405$

Now $f(x_4) = f(2.7405) = 0.000001$

Hence the root is 2.7405.

1.5 Newton – Raphson Method of Solving a Nonlinear Equation

The Newton – Raphson method, or Newton Method, is a powerful technique for solving equations numerically. Like so much of the differential calculus, it is based on the simple idea of linear approximation. The Newton Method, properly used, usually homes in on a root with devastating efficiency. The Newton-Raphson method is based on the principle that if the initial guess of the root of $f(x)=0$ is at x_i, then if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root.

Using the definition of the slope of a function, at $x = x_i$

$$f'(x_i) = \tan\theta$$

$$= \frac{f(x_i)-0}{x_i - x_{i+1}},$$

which gives

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1)$$

Equation (1) is called the Newton-Raphson formula for solving nonlinear equations of the form $f(x)=0$. One can repeat this process until one finds the root within a desirable tolerance. The order of convergence of N-R method is quadratic.

1.5.1 Newton's algorithm for finding the P^th^ root of a Number N:

The P^{th} root of the a positive number N is the root of the equation.

Let $x = N^{\frac{1}{p}}$

$$x^p - N = 0$$

$$f(x) = x^p - N$$

$$f^1(x) = px^{p-1}$$

By Newton's algorithm

$$x_k = 1 - \frac{f(x)}{f'(x)}$$

$$x_k = \left(\frac{x_k^P - N}{Px_k^{p-1}}\right) + 1$$

$$= \frac{Px_k{}^p - x_k^P - N}{Px_{k^{p-1}}}$$

$$= \frac{(p-1)x_k^P + N}{Px_k^{p-1}}$$

1.5.2 Newton Raphson formula for cube root of a positive number k.

$$x = \sqrt[3]{k}$$

$$f(x) = x^3 - k = 0$$

$$f^1(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{x_n^3 - k}{3x_n^2}$$

$$= \frac{1}{3}\left[2x_n + \frac{k}{x_n^2}\right]$$

1.5.3 Newtons – Raphson – formula for $\sqrt{a}$ or $\sqrt{N}$ Or

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ n = 0, 1, 2, …..

Let

$x = \sqrt{a}$

$x^2 - a = 0$

Let $f(x) = x^2 - a$

$f'(x) = 2x$

By Newton – Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - a}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + a}{2x_n}$$

1.5.4 When should we not use Newton – Raphson method:

If x_1 is the exact root and x_0 is its approximate value of the equation f(x) =0. We know that $x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$. If $f^1(x_0)$ is small, the error $\frac{f(x_0)}{f^1(x_0)}$ will be large and the computation of the root by this method will be a slow process. Hence the method should not be used in case where the graph of the functions when it crosses the x axis is nearly horizontal.

Example 1: Evaluate $\sqrt{12}$ applying Newton formula.

Solution:

Let x = $\sqrt{12}$

$x^2 = 12$; $x^2 - 12 = 0$

$f(x) = x^2 - 12$

f(3) = -ve,

f(4) = +ve

take $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.5 - \frac{f(3.5)}{f'(3.5)} = 3.464$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.464 - \frac{f(3.464)}{f'(3.464)} = 3.464$$

Hence the root of the equation is 3.464.

Example 2: Iteration formula to find the reciprocal of a positive number N by Newton Raphson method

Solution:

Let $x = \frac{1}{N}$

$$N = \frac{1}{x} \Rightarrow \frac{1}{x} - N = 0$$

Now $f(x) = \frac{1}{x} - N$

$f'(x) = -\frac{1}{x^2}$

By Newton's Formula

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \left(\frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}\right)$

$= x_n + \left(\frac{1}{x_n} - N\right) x_n^2$

$x_{n+1} = x_n(2 - Nx_n)$

Example 3: Find the square root of 8. by Newton – Raphson Method.

Solution:

Given N = 8 Clearly $2 < \sqrt{8} < 3$ taking $x_0 = 2.5$ we get

$x_1 = \frac{1}{2}\left[x_0 + \frac{N}{x_0}\right] = \frac{1}{2}\left[2.5 + \frac{8}{2.5}\right] = 2.85$

$x_2 = \frac{1}{2}\left[x_1 + \frac{N}{x_1}\right] = \frac{1}{2}\left[2.85 + \frac{8}{2.85}\right] = 2.8285$

$x_3 = \frac{1}{2}\left[x_2 + \frac{N}{x_2}\right] = \frac{1}{2}\left[2.828 + \frac{8}{2.828}\right] = 2.8284$

$x_4 = \frac{1}{2}\left[x_3 + \frac{N}{x_3}\right] = \frac{1}{2}\left[2.8284 + \frac{8}{2.8284}\right] = 2.8284$

$\therefore \sqrt{8} = 2.8284$

Example 4: By applying Newton's method twice, find the real root near 2 of the equation $x^4 - 12x + 7 = 0$

Solution:

Let $f(x) = x^4 - 12x + 7$

$f(x) = 4x^3 - 12$

Put $x_0 = 2$, $f(x_0) = -1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2.05$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.05 - \frac{f(2.05)}{f'(2.05)} = 2.6706$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.6706 - \frac{f(2.6706)}{f'(2.6706)} = 2.6706$

Therefore the root of the equation is 2.6706.

Example 5: If an approximate root of the equation $x(1 - logx) = 0.5$ lies between 0.1 and 0.2 find the value of the root correct to three decimal places.

Solution:

Given f(x) = x(1- log x) -0.5

$f^1(x) = (1- \log x) + x \left(-\frac{1}{x}\right)$

= - log x

f(0.1) = 0.1 [1-log (0.1)] – 0.5 = 0.1697 (-ve)

f(0.2) = 0.2 [1-log (0.2)] – 0.5 = 0.02188 (+ve)

$x_0 = 0.9$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$= 0.2 - \frac{0.2(1 - \log(0.2) - 0.5}{-\log(0.2)}$$

$$= 0.2 - \frac{0.02188}{1.6094} = 0.1864$$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$$= 0.1864 - \frac{0.1864(1 - \log(0.1864) - 0.5}{-\log(0.1864)}$$

$$= 0.1864 + \frac{0.0004666}{1.6799} = 0.1866$$

$$x_3 = x_1 - \frac{f(x_2)}{f^1(x_2)}$$

$$= 0.1866 - \frac{0.1866(1 - \log(0.1866) - 0.5}{-\log(0.1866)}$$

$$= 0.1866$$

Hence the approximate root is 0.1866.

Example 6: Find the approximate root of $xe^x = 3$ by Newton's Raphson method correct to three decimal places.

Solution:

Given $f(x) = xe^x - 3$

$F^I(x) = xe^x + e^x$

$f(1) = 1e^{-1} - 3 = 2.7182 - 3 = - 0.2817$ (-ve)

$f(1.5) = 1.5e^{1.5} - 3 = 3.7223$ (+ve)

Here f(1) is –ve (Negative) and f(1.5) is +ve (positive). Therefore the root lies between 1 ad 1.5. Since the magnitude of f(1) < f(1.5) we can take the initial approximate $x_0 = 1$.

The first approximation is

$$x_1 = x_0 \frac{f(x_0)}{f^1(x_0)}$$

$$= 1 - \frac{-0.2817}{5.4363} = 1.0518$$

The second approximation

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$$= 1.0518 - \frac{-0.0111}{5.8739}$$

$$= 1.0499$$

The third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)}$$

$$= 1.0499 - \frac{1.0499\, e^{1.0499} - 3}{1.0499\, e^{1.0499} + e^{1.04999}}$$

$$= 1.0499$$

Hence the root of xe^x is 1.0499

Example 7: Using Newton Raphson method to find correct to four decimals the root between 0 and 1 of the equation $x^3 - 6x + 4 = 0$.

Solution:

Given $f(x) = x^3 - 6x + 4$

$f(0) = 4, \quad f(1) = -1$

$f(0)\, f(1) = 4(-1) < 0$

The root of f (x) = 0 lies between 0 and 1 the value of the root is near to 1. Let $x_0 =$ 0.7 an approximate

$$f(x) = x^3 - 6x + 4, \quad f^1(x) = 3x^2 - 6$$

$$f(0) = x_0^3 - 6x_0 + 4, \; f^1(x_0) = 3x_0^2 - 6$$

$$f(0-7) = (0.7)^3 - 6(0.7) + 4 = 0.143$$

$$f^1(0.7) = 3(0.7)^2 - 6 = -4.53$$

Then by Newton's iteration formula we get

$$x_1 = x_0 - \frac{f(x)}{f^1(x_0)}$$

$$= 0.7316$$

$$f(x_1) = 0.0019805$$

$$f^1(x_1) = 3 \times (0.7316)^2 - 6 = -4.394$$

The second approximate

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_2)}$$

$$= 0.7316 + \frac{0.0019805}{4.39428}$$

$$= 0.73250699$$

$$= 0.7321$$

The root of the equation = 0.7321

Lets sum up

Numerical methods are techniques used to find approximate solutions to equations that cannot be solved analytically. Here is a brief overview of four common methods to the learners:

❖ **Bisection Method**

Concept:

- The bisection method is a straightforward approach that repeatedly bisects an interval and then selects a subinterval in which a root must lie. This process is repeated until the interval is sufficiently small.

Advantages:

- Simple and robust.
- Guaranteed to converge if the initial interval is chosen correctly.

Disadvantages:

- Convergence can be slow.
- Requires the function to change sign over the interval.

❖ **Iteration Method (Fixed-Point Iteration)**

Concept:

- This method transforms an equation of the form f(x)=0 into x=g(x) and uses an iterative process to converge to a solution. It relies on the function g(x) being chosen such that the sequence of iterations converges to a fixed point.

Advantages:

- Simple to implement.
- Effective for well-behaved functions.

Disadvantages:

- Convergence is not guaranteed if the function g(x) is not chosen correctly.
- May be slow to converge.

❖ **Regula Falsi Method (False Position Method)**

Concept:

- This method uses linear interpolation to approximate the root of an equation. It refines the interval in which the root lies by repeatedly applying a linear formula to find closer approximations.

Advantages:

- Typically faster than the bisection method.
- More reliable convergence than fixed-point iteration.

Disadvantages:

- Can be slower than other methods for certain types of functions.
- May converge slowly if the function is not linear near the root.

❖ **Newton-Raphson Method**

Concept:

- This method uses the derivative of the function to find better approximations of the root. It iteratively refines the guess using the tangent line at the current approximation.

Advantages:

- Typically converges very quickly if the initial guess is close to the actual root.
- Efficient for well-behaved functions.

Disadvantages:

- Requires the calculation of the derivative.
- May fail to converge if the initial guess is not close to the root or if the function is not well-behaved.

Conclusion

Each of these methods has its strengths and weaknesses. The choice of method depends on the specific problem, the nature of the function, and the required precision. Understanding these methods provides a solid foundation for tackling a wide range of algebraic and transcendental equations numerically.

Self Assessment Questions:

1. Using bisection method, find the negative root of $x^3 - 4x + 9 = 0$.
2. Assuming that a root of $x^3 - 9x + 1 = 0$ lies in the interval (2,4), find that root by bisection method.
3. Solve for x from $\cos x - xe^x = 0$ by iteration method.
4. Solve $x^3 = 2x + 5$ for the positive root by iteration method.
5. Solve for a positive root of $x - \cos x = 0$ by Regula Falsi method.
6. Solve the equation $x\, tanx = -1$ by Regula Falsi method starting with a = 2.5 and b = 3 correct to three decimal places.
7. Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method.
8. Solve for positive root by Newton's method of $2x - \log_{10} x = 7$.

Answers for check-up your progress:

1. -2.7064, 2. 2.9429, 3. 0.5177, 4. 2.0945, 5. 0.7391, 6. 2.798, 7. 0.6071, 8. 3.7893

References:

1. P. Kandasamy, K. Thilagavathy, K. Gunavathi, Numerical Methods, S. Chand and Co. New Delhi.
2. B. S. Grewal, Numerical Methods in Engineering and Science with Programs in C, C++ and MATLAB, Khanna Publishers.
3. S.R.K. Iyengar, R. K. Jain, Numerical Methods, New Age International Publishers.

E-Resources

1. http://www.assakkaf.com/Courses/ENCE203/Lectures/Chapter4d.pdf
2. https://www.lnjpitchapra.in/wp-content/uploads/2020/05/file_5eb6536f2e4f2.pdf
3. https://www3.nd.edu/~zxu2/acms40390F12/Lec-2.2.pdf

Unit 2

Numerical Solutions of Equations

2.0 Introduction

In the field of Science and Engineering, the solution of equations of the form $f(x) = 0$ occurs in many applications. If $f(x)$ is a polynomial of degree two or three or four, exact formulae are available. But, if $f(x)$ is a transcendental function like $a + be^x + c\sin x + d\log x$ etc., the solution is not exact and we do not have formulae to get the solutions. When the coefficients are numerical values, we can adopt various numerical approximate methods such as,

1. Generalised Newton's Method
2. Ramanujan's Method
3. The Secant Method
4. Muller's Method
5. Graeffe's Root squaring Method

2.1 Generalised Newton's Method

If α is a root of $f(x) = 0$ with multiplicity p, then the iteration formula will be

$$x_{n+1} = x_n - p\frac{f(x_n)}{f'(x_n)}$$

This means that $\frac{1}{p}f'(x_n)$ is the slope of the line through (x_n, y_n) and intersecting the axis of x at $(x_{n+1}, 0)$. Since α is a root of $f(x) = 0$ with multiplicity p, it implies that α is also a root of $f'(x) = 0$ with multiplicity (p-1) and it is a root of $f''(x) = 0$ with multiplicity (p-2) and so on. Therefore

$$x_0 - p\frac{f(x_0)}{f'(x_0)}, x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)}, x_n - (p-2)\frac{f''(x_0)}{f'''(x_0)}$$

will all have the same value if the initial approximation x_0 is chosen close to the actual root. The order of convergence of this method is two.

Example 1: Find the double root of the equation $x^3 - x^2 - x + 1 = 0$.

Solution:

Let $f(x) = x^3 - x^2 - x + 1$

$f'(x) = 3x^2 - 2x - 1$

$f''(x) = 6x - 2$

Starting with $x_0 = 0.9$, we have

$$x_1 = x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.9 - 2\frac{f(0.9)}{f'(0.9)} = 0.9 - \frac{2 * 0.019}{-0.37} = 1.003$$

$$x_1 = x_0 - (2-1)\frac{f'(x_0)}{f''(x_0)} = 0.9 - \frac{f'(0.9)}{f''(0.9)} = 0.9 - \frac{(-0.37)}{3.4} = 1.009$$

The closeness of these values implies that there is a double root near $x = 1$.

Choosing $x_1 = 1.01$ for the next approximation, we get

$$x_2 = x_1 - 2\frac{f(x_1)}{f'(x_1)} = 1.01 - 2\frac{f(1.01)}{f'(1.01)} = 1.01 - \frac{2 * 0.0002}{0.0403} = 1.0001$$

$$x_2 = x_1 - (2-1)\frac{f'(x_1)}{f''(x_1)} = 1.01 - \frac{f'(1.01)}{f''(1.01)} = 1.01 - \frac{0.0403}{4.06} = 1.0001$$

This shows that there is a double root at x = 1.0001 which is quite near the actual root x = 1.

Example 2: Find the double root of $x^3 - 5.4x^2 + 9.24x - 5.096 = 0$ given that it is nearer to 1.5.

Solution:

Let $f(x) = x^3 - 5.4x^2 + 9.24x - 5.096$

$f'(x) = 3x^2 - 10.8x + 9.24$

$f''(x) = 6x - 10.8$

Starting with $x_0 = 1.5$, we have

$$x_1 = x_0 - 2\frac{f(x_0)}{f'(x_0)} = 1.5 - 2\frac{f(1.5)}{f'(1.5)} = 1.3954$$

$$x_1 = 1.5 - (2-1)\frac{f'(1.5)}{f''(1.5)} = 1.5 - \frac{f'(1.5)}{f''(1.5)} = 1.3952$$

The closeness of these values implies that there is a double root near $x = 1.4$

Proceeding in this way we get

$x_2 = 1.3966, x_3 = 1.4024, x_4 = 1.4211, ...$

The root is approximately 1.4 correct to one decimal place.

2.2 Ramanujan's Method

It is an iterative method which can be used to determine the smallest real root of $f(x) = 0$. Where $f(x) = 1 - (a_1 x + a_2 x^2 + \cdots)$, here $a_1, a_2, a_3, ...$ are real constants. For small values of x, we can arrive

$[1-(a_1x+a_2x^2+\cdots)]^{-1}=b_1+b_2x+b_3x^2+\cdots$

$1+(a_1x+a_2x^2+\cdots)+(a_1x+a_2x^2+\cdots)^2+\cdots=b_1+b_2x+b_3x^2+\cdots$

Comparing the coefficients of like power of x on both sides, we obtain

$b_1=1$

$b_2=a_1=a_1.1=a_1b_1$

$b_3=a_2+a_1^2=a_2b_1+a_1b_2$

And so on

$b_n=a_1b_{n-1}+a_2b_{n-2}+\cdots+a_{n-1}b_1$ where n=2,3,4,…

The ratios $\frac{b_{i-1}}{b_i}$ called the convergents, approach in the limit, the smallest root of $f(x)=0$.

Example 1: Using Ramanujan's method obtain first five convergence of the equation $f(x)=x^3-6x^2+11x-6$.

Solution:

Given $f(x)=x^3-6x^2+11x-6$

Let $x^3-6x^2+11x-6=0$

$\frac{x^3-6x^2+11x}{6}-1=0$

The smallest real root of the given equation is

$$\left[1-\left(\frac{x^3-6x^2+11x}{6}\right)\right]^{-1}=b_1+b_2x+b_3x^2+b_4x^3+b_5x^4+b_6x^5+b_7x^6+\cdots$$

Here $a_1=\frac{11}{6}, a_2=-1, a_3=\frac{1}{6}, a_4=0, a_5=0, a_6=0, a_7=0$

$b_1=1$

$b_2=a_1b_1=\frac{11}{6}*1=\frac{11}{6}$

$b_3=a_2b_1+a_1b_2=-1*1+\frac{11}{6}+\frac{11}{6}=\frac{85}{36}$

$b_4=a_1b_3+a_2b_2+a_3b_1=\frac{11}{6}*\frac{85}{36}+\ (-1)*\frac{11}{6}+\frac{1}{6}*1=\frac{575}{216}$

$b_5=a_1b_4+a_2b_3+a_3b_2+a_4b_1=\frac{11}{6}*\frac{575}{216}+\ (-1)*\frac{85}{36}+\frac{1}{6}*\frac{11}{6}+0=\frac{3675}{1296}$

$b_6=a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1=\frac{11}{6}*\frac{3675}{1296}+(-1)*\frac{575}{216}+\frac{1}{6}*\frac{85}{36}=\frac{22785}{7776}$

1st Convergent $\frac{b_1}{b_2}=0.5454$

2nd Convergent $\frac{b_2}{b_3}=0.7764$

3rd Convergent $\frac{b_3}{b_4} = 0.8869$

4th Convergent $\frac{b_4}{b_5} = 0.9387$

5th Convergent $\frac{b_5}{b_6} = 0.9670$

The smallest root for the given equation is 0.967.

Example 2: Find a root of the equation $x\, e^x = 1$ by Ramanujan's method.

Solution:

Given $f(x) = x\, e^x - 1$

Expanding e^x in ascending powers of x and simplifying, we get

$1 = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \cdots$

Here $a_1 = 1, a_2 = 1, a_3 = \frac{1}{2}, a_4 = \frac{1}{6}, a_5 = \frac{1}{24}, \ldots$

Then $b_1 = 1$

$b_2 = a_1 b_1 = 1$

$b_3 = a_2 b_1 + a_1 b_2 = 1 + 1 = 2$

$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = 2 + 1 + \frac{1}{2} = \frac{7}{2}$

$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 = \frac{7}{2} + 2 + \frac{1}{2} + \frac{1}{6} = 6.1667$

$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 = 10.8750$

Hence

1st Convergent $\frac{b_1}{b_2} = 1$

2nd Convergent $\frac{b_2}{b_3} = 0.5$

3rd Convergent $\frac{b_3}{b_4} = 0.5714$

4th Convergent $\frac{b_4}{b_5} = 0.5676$

5th Convergent $\frac{b_5}{b_6} = 0.5670$

The smallest root for the given equation is 0.567.

2.3 The Secant Method

This method is an improvement over the method of false position as it does not require the condition $f(x_0)f(x_1) < 0$ of that method. Here also the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find the next approximation. Also it is not necessary that the interval must contain the root. Taking x_0, x_1 as the initial limits of the interval, we write the equation of the chord joining these as

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}\,(x - x_1)$$

Then the abscissa of the point where it crossed the x-axis (y=0) is given by

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}\,f(x_1)$$

Which is an approximation to the root. The general formula for successive approximations is given by

$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\,f(x_n), n \geq 1.$

The rate of convergence of this method is 1.6 which is faster than that of the method of false position.

Example 1: Find a root of the equation $x^3 - 2x - 5 = 0$ using secant method correct to three decimal places.

Solution:

Let $f(x) = x^3 - 2x - 5$

Here $f(2) = -1\ and\ f(3) = 16$

Hence the root lies between 2 and 3.

Taking initial approximations $x_0 = 2\ and\ x_1 = 3$, by secant method, we have

$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}\,f(x_1) = 3 - \frac{(3-2)}{(16+1)} * 16 = 2.0588$

$f(x_2) = -0.3908$

$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)}\,f(x_2) = 2.0813$

$f(x_3) = -0.1472$

$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)}\,f(x_3) = 2.0948$

$f(x_4) = 0.00304$

$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)}\,f(x_4) = 2.0945$

Hence the root is 2.094 correct to 3 decimal places.

Example 2: Find a root of the equation $xe^x = \cos x$ using secant method correct to three decimal places.

Solution:

Let $f(x) = xe^x - \cos x$

Here $f(0) = 1 \; and \; f(1) = -2.17798$

Hence the root lies between 0 and 1

Taking initial approximations $x_0 = 0 \; and \; x_1 = 1$, by secant method, we have

$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 1 + \frac{-2.17798}{3.17798} = 0.31467$

$f(x_2) = 0.51987$

$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 0.4467$

$f(x_3) = 0.20354$

$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 0.5317$

Repeating this process, the successive approximations are $x_5 = 0.5169, x_6 = 0.5177, x_7 = 0.5177$.

Hence the root is 0.5177.

2.4 Muller's Method

This method is a generalization of the secant method as it doesn't require the derivative of the function. It is an iterative method that requires three starting points. Here $y = f(x)$ is approximated by a second degree parabola passing through these three point $(x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1}) \; and \; (x_i, y_i)$ in the vicinity of the root. Then the root of this quadratic is taken as the next approximations x_{i+1} to the root of $f(x) = 0$.

Assuming the equation of the parabola through the points $(x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1}) \; and \; (x_i, y_i)$ to be

$$y = ax^2 + bx + c$$

$$y_{i-2} = ax_{i-2}{}^2 + bx_{i-2} + c$$

$$y_{i-1} = ax_{i-1}{}^2 + bx_{i-1} + c$$

$$y_i = ax_i{}^2 + bx_i + c$$

Eliminating a, b, c we get

$$y = \frac{(y_{i-2}\lambda_i + y_{i-1}\delta_i + y_i)\lambda_i \lambda^2}{\delta_i} + \frac{y_{i-2}\lambda_i^2 - y_{i-1}\delta_i^2 + y_i(\lambda_i + \delta_i)}{\delta_i}\lambda + y_i$$

Where $\lambda = \frac{x - x_i}{x_i - x_{i-1}}$, $\lambda_i = \frac{x_i - x_{i-1}}{x_{i-1} - x_{i-2}} \; and \; \delta_i = \frac{x_i - x_{i-2}}{x_{i-1} - x_{i-2}}$

Now to find a better approximation to the root, we need the unknown quantity λ.

Since,

$\mu_i = y_{i-2}\lambda_i^2 - y_{i-1}\delta_i^2 + y_i(\lambda_i + \delta_i)$

Example 1: Apply Muller's method to find the root of the equation $\cos x = xe^x$ which lies between 0 and 1.

Solution:

Let $y = \cos x - xe^x$

Taking the initial approximations as

$x_{i-2} = -1, x_{i-1} = 0, x_i = 1$

$y_{i-2} = \cos 1 + e^{-1}, y_{i-1} = 1, y_i = cos\ 1 - e$

$\lambda = x - 1, \lambda_i = 1, \delta_i = 2$

$\mu_i = (\cos 1 + e^{-1}) - 4 + 3\ (\cos 1 - e)$

We get two values of λ^{-1}.

We choose the negative sign so that the numerator is largest in magnitude and obtain $\lambda = -0.5585$.

The next approximation to the root is given by

$x_{i+1} = x_i + \lambda(x_i - x_{i-1}) = 1 - 0.5585 = 0.4415.$

Repeating the above process, we get

$x_{i+2} = 0.5125, x_{i+3} = 0.5177, x_{i+4} = 0.5177$

Hence the root is 0.5177.

2.5 Graeffe's Root Squaring Method

This method has an advantage over the other methods that it does not require any prior information about the root. But it is applicable to polynomial equations only and is capable of giving all the roots. Consider the polynomial equation

$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ (1)

Separating the even and odd powers of x and squaring, we get

$(x^n + a_2x^{n-2} + a_4x^{n-4} + \cdots)^2 = (a_1x^{n-1} + a_3x^{n-3} + \cdots)^2$

Putting $x^2 = y$ and simplifying,

$y^n + b_1y^{n-1} + \cdots + b_{n-1}y + b_n = 0$ (2)

Where $b_1 = -a_1^2 + 2a_2$

$b_2 = a_2^2 - 2a_1a_3 + 2a_4$

…………………………

$b_n = (-1)^n \alpha_n^2$

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ be the roots of (1) then the roots of (2) are $\alpha_1^2, \alpha_2^2, \alpha_3^2, \dots, \alpha_n^2$.

After m squaring, let the new transferred equation be

$z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n = 0$

Whose roots $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n$ are such that $\gamma_i = \alpha_i^{2m}, i = 12, \dots n$

Assuming that $\frac{|\gamma_2|}{|\gamma_1|} = \frac{\gamma_2}{\gamma_1}, \dots, \frac{|\gamma_n|}{|\gamma_{n-1}|} = \frac{\gamma_n}{\gamma_{n-1}}$ are negligible as compared to unity. Also γ_i being an even power of α_i is always positive.

We have $\sum \gamma_1 = -c_1$ that is $c_1 = -\gamma_1 \left(1 + \frac{\gamma_2}{\gamma_1} + \frac{\gamma_3}{\gamma_1} + \cdots\right)$

$\sum \gamma_1 \gamma_2 = c_2$ that is $c_2 = \gamma_1 \gamma_2 \left(1 + \frac{\gamma_3}{\gamma_1} + \cdots\right)$

$\sum \gamma_1 \gamma_2 \gamma_3 = -c_3$ that is $c_3 = -\gamma_1 \gamma_2 \gamma_3 \left(1 + \frac{\gamma_4}{\gamma_1} + \cdots\right)$

………………………..

We get $\gamma_1 = -c_1, \gamma_2 = -\frac{c_2}{c_1}$, $\gamma_3 = -\frac{c_3}{c_2}$, …

Now $\gamma_i = \alpha_i^{2m}$ hence $\alpha_i = (\gamma_i)^{1/2m}$

Thus we can find $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ the roots of given equation.

Example 1: Find all the roots of the equation $x^3 - 2x^2 - 5x + 6 = 0$ by Graeffe's method, squaring thrice.

Solution:

Let $f(x) = x^3 - 2x^2 - 5x + 6$ (1)

By Descart's rule of signs, ther being two changes of sign, then f(x) has two positive roots. Also

$f(-x) = -x^3 - 2x^2 + 5x + 6$

f(-x) has one change in sign, there is one negative root.

Rewriting (1) as $x^3 - 5x = 2x^2 - 6$ and squaring we get $y(y-5)^2 = (2y-6)^2$ where $y = x^2$

$y(y^2 + 49) = 14y^2 + 36$

Squaring again and putting $y^2 = z$, we obtain $z(z+49)^2 = (14z + 36)^2$

$z(z^2 + 1393) = 98z^2 + 1296$

Squaring once again and putting $z^2 = u$, we get $u(u + 1393)^2 = (98u + 1296)^2$

$u^3 - 6818u^2 + 1686433u - 1679616 = 0$ (2)

If the roots of (2) are $\gamma_1, \gamma_2, \gamma_3$ then $\gamma_1 = -c_1 = 6818$

$\gamma_2 = -\frac{c_2}{c_1} = 247.3501$

$\gamma_3 = -\frac{c_3}{c_2} = 0.996$

If $\alpha_1, \alpha_2, \alpha_3$ be the roots of (1), then

$\alpha_1 = \gamma_1^{\frac{1}{8}} = 3.0144$, $\alpha_2 = \gamma_2^{\frac{1}{8}} = 1.9914$, $\alpha_3 = \gamma_3^{\frac{1}{8}} = 0.9995$

Hence the roots are 3, -2, 1.

Lets Sum up

Numerical methods for solving nonlinear equations are essential when analytical solutions are not feasible. Here is an overview of some advanced methods: Generalised Newton's Method, Ramanujan's Method, The Secant Method, Muller's Method, and Graeffe's Root Squaring Method.

❖ **Generalised Newton's Method**

Concept:

- An extension of the classic Newton-Raphson method, the Generalised Newton's Method is designed to handle systems of nonlinear equations. It iteratively refines guesses by solving a linear system derived from the Jacobian matrix of the functions.

Advantages:

- Efficient for systems of equations.
- Quadratic convergence near the root.

Disadvantages:

- Requires computation of the Jacobian matrix.
- Can be complex to implement and may not converge if the initial guess is not close to the true solution.

❖ **Ramanujan's Method**

Concept:

- Based on the work of the famous mathematician Srinivasa Ramanujan, this method is less well-known but involves iterative processes often based on series expansions or other ingenious approximations to solve equations.

Advantages:

- Can provide very accurate results for specific types of equations.

- Often involves elegant and novel approaches.

Disadvantages:

- Not widely documented or used compared to other methods.
- May be more difficult to apply to general problems.

❖ **The Secant Method**

Concept:

- An improvement over the bisection method, the Secant Method uses secant lines (instead of tangents as in Newton-Raphson) to approximate the root. It requires two initial guesses but does not need the calculation of derivatives.

Advantages:

- Faster convergence than bisection.
- Does not require derivative calculation.

Disadvantages:

- Slower convergence than Newton-Raphson.
- May fail to converge if the initial guesses are not well chosen.

❖ **Muller's Method**

Concept:

- This method uses quadratic interpolation to approximate the root of a function. It extends the idea of the Secant Method by fitting a parabola through three points and finding the root of the interpolating quadratic polynomial.

Advantages:

- Can converge faster than the Secant and Newton-Raphson methods.
- Effective for complex roots and well-suited for polynomial equations.

Disadvantages:

- More computationally intensive.
- Requires careful handling of complex arithmetic.

❖ **Graeffe's Root Squaring Method**

Concept:

- This method is used primarily for finding all the roots of a polynomial. It repeatedly squares the polynomial and separates the roots by squaring the magnitude of each root.

Advantages:

- Efficient for finding all roots of a polynomial.
- Can handle large polynomials with multiple roots.

Disadvantages:

- May require high precision arithmetic to avoid numerical instability.
- Not suitable for non-polynomial equations.

Conclusion

Each method has its unique strengths and is suited for different types of problems. Understanding these methods expands the toolkit for solving a wide range of nonlinear equations, enabling the selection of the most appropriate method based on the problem's characteristics.

Self Assessment Questions:

1. Find the smallest root, correct to 4 decimal places of the equation $f(x) = 3x - \cos x - 1 = 0$ by Ramanujan Method.
2. Find the smallest root, correct to 4 decimal places of the equation $\sin x = x - \frac{1}{2}$, by Ramanujan Method.
3. Find a root of the equation $x^3 + x^2 + x + 7 = 0$ using secant method correct to three decimal places.
4. Find a root of the equation $x\ log_{10}x = 1.9$ using secant method correct to three decimal places.
5. Apply Muller's method to find the root of the equation $x^3 - 2x - 1 = 0$.
6. Apply Muller's method to find the root of the equation $\log x = x - 3$, taking $x_0 = 0.25, x_1 = 0.5\ and\ x_2 = 1$
7. Apply Graeffe's method to find all the roots of the equation $x^4 - 3x + 1 = 0$.
8. Determine all the roots of the equation $x^3 - 9x^2 + 18x - 6 = 0$ by Graeffe's method.

Answers for check-up your progress

1. 0.6071, 2. 1.4973, 3. -2.0625, 4. 3.496, 5. 2.26, 6. 3.14, 7. 1.1892, 0.3379, -0.7636 ± 1.381i, 8. 6.3, 2.3, 0.4

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Unit 3

FINITE DIFFERENCES

3.0 Introduction

Let us assume that values of a function $y=f(x)$ are known for a set of equally spaced values of x given by $\{x_0,x_1,...,x_n\}$, such that the spacing between any two consecutive values is equal. Thus, $x_1=x_0+h$, $x_2=x_1+h$, ..., $x_n=x_{n-1}+h$, so that $x_i=x_0+ih$ for $i=1,2,...,n$. We consider two types of differences known as forward differences and backward differences of various orders. These differences can be tabulated in a finite difference table as explained in the sub sequent sections.

3.1 Forward Difference

Let $y_0,y_1,...,y_n$ be the values of a function $y=f(x)$ at the equally spaced values of $x=x_0,x_1,...,x_n$. The differences between two consecutive y given by y_1-y_0, $y_2-y_1,...,\ y_n-y_{n-1}$ are called the first order forward differences of the function $y=f(x)$ at the points $x_0,x_1,...,x_{n-1}$. These differences are denoted by,

$$\Delta y_0 = y_1 - y_0\ , \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$$

Where Δ is termed as the forward difference operator defined by,

$\Delta f(x) = f(x + h) - f(x)$

Thus, $\Delta y_i = y_{i+1} - y_i$, for $i = 0, 1, 2, ..., n - 1$, are the first order forward differences at xi.

The differences of these first order forward differences are called the second order forward differences.

Thus,

$\Delta^2 y = \Delta(\Delta y) = \Delta y_i + 1 - \Delta y_i\ , for\ i = 0, 1, 2, \ldots, n-2$

Evidently,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

And, $\Delta^2 y_i = y_{i+2} - y_{i+1} - (y_{i+1} - y_i)$

Finally, we can define the nth order forward difference by,

$$\Delta^n y_0 = y_n - ny_{n-1} + \frac{n(n-1)}{2} y_{n-2} + \cdots + (-1)^n y_0$$

The forward differences of various orders for a table of values of a function $y=f(x)$, are usually computed and represented in a diagonal difference table. A diagonal difference table for a table of values of *y=f*(*x*), for six points $x_0, x_1, x_2, x_3, x_4, x_5$ is shown here.

Diagonal difference Table for *y=f*(*x*):

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
0	x_0	y_0					
			Δy_0				
1	x_1	y_1		$\Delta^2 y_0$			
			Δy_1		$\Delta^3 y_0$		
2	x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
			Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
3	x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
			Δy_3		$\Delta^3 y_2$		
4	x_4	y_4		$\Delta^2 y^3$			
			Δy_4				
5	x_5	y_5					

The entries in any column of the differences are computed as the differences of the entries of the previous column and one placed in between them. The upper data in a column is subtracted from the lower data to compute the forward differences. We notice that the forward differences of various orders with respect to y_i are along the forward diagonal through it. Thus $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0 \; and \; \Delta^5 y_0$ lie along the top forward diagonally through y_0.

Example 1: Given the table of values of $y = f(x)$

X	1	3	5	7	9
y	8	12	21	36	62

Form the diagonal difference table and find the values of $\Delta f(5), \Delta^2 f(3), \Delta^3 f(1)$.

Solution:

The diagonal difference table is,

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0	1	8				
			4			
1	3	12		5		
			9		1	
2	5	21		6		4
			15		5	
3	7	36		11		
			26			
4	9	62				

From the table, we find that $\Delta f(5) = 15$, the entry along the diagonal through the entry 21 of f(5). Similarly, $\Delta^2 f(3) = 6$ the entry along the diagonal through f(3). Finally, $\Delta^3 f(1) = 1$.

3.2 Backward Difference

The backward differences of various orders for a table of values of a function $y = f(x)$ are defined in a manner similar to the forward differences. The backward difference operator ∇ is defined by $\nabla f(x) = f(x) - f(x-h)$

Thus $\nabla \mathrm{y_k} = y_k - y_{k-1}$ for k=1,2,…n

That is $\nabla \mathrm{y}_1 = y_1 - y_0, \nabla \mathrm{y}_2 = y_2 - y_1, \dots, \nabla \mathrm{y}_n = y_n - y_{n-1}$

The backward differences of second order are defined by

$$\nabla^2 \mathrm{y_k} = \nabla y_k - \nabla y_{k-1} = y_k - 2y_{k-1} + y_{k-2}$$

Hence $\nabla^2 \mathrm{y}_2 = y_2 - 2y_1 + y_0$ and $\nabla^2 \mathrm{y}_n = y_n - 2y_{n-1} + y_{n-2}$

Higher order backward differences can be defined in a similar manner.

Thus $\nabla^3 \mathrm{y}_n = y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}$, etc.

The backward differences of various orders can be computed and placed in a diagonal difference table. The backward differences at a point are then found along the backward diagonal through the point. The following table shows the backward differences entries.

Diagonal difference table of backward differences

i	x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$	$\nabla^5 y_i$
0	x_0	y_0					
			∇y_1				
1	x_1	y_1		$\nabla^2 y_2$			
			∇y_2		$\nabla^3 y_3$		
2	x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
			∇y_3		$\nabla^3 y_4$		
3	x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
			∇y_4		$\nabla^3 y_5$		
4	x_4	y_4		$\nabla^2 y_5$			
			∇y_5				
5	x_5	y_5					

The entries along a column in the table are computed as the differences of the entries in the previous column and are placed in between. We notice that the backward differences of various orders with respect to y_i are along the backward diagonal through it. Thus $\nabla y_5, \nabla^2 y_5, \nabla^3 y_5, \nabla^4 y_5$ *and* $\nabla^5 y_5$ lie along the lowest backward diagonally through y_5.

Example 1: Given the table of values of $y = f(x)$

X	1	3	5	7	9
y	8	12	21	36	62

Find the values of $\nabla y_7, \nabla^2 y_9, \Delta^3 y_9$.

Solution:

The diagonal difference table is,

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0	1	8				
			4			
1	3	12		5		
			9		1	
2	5	21		6		4
			15		5	
3	7	36		11		
			26			
4	9	62				

From the table, we can easily find, $\nabla y_7 = 15, \nabla^2 y_9 = 11, \Delta^3 y_9 = 5$.

Example 2: Find the 7th term of the sequence 2, 9, 28, 65, 126, 217 and also find the general term.

Solution:

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2				
		7			
1	9		12		
		19		6	
2	28		18		0
		37		6	
3	65		24		0
		61		6	
4	126		30		
		91			
5	217				

7th term$=y_6 = (1+\Delta)^6\, y_0 = y_0 + 6C_1\, \Delta y_0 + 6C_2\, \Delta^2 y_0 + 6C_3\, \Delta^3 y_0 + \cdots$

= 2+6(7)+15(12)+20(6)+0 = 344

$y_n = (1+\Delta)^n\, y_0 = y_0 + nC_1\, \Delta y_0 + nC_2\, \Delta^2 y_0 + nC_3\, \Delta^3 y_0 + \cdots$

$$= 2 + n(7) + \frac{n(n-1)}{2}(12) + \frac{n(n-1)(n-2)}{6}(6) + 0$$
$$= n^3 + 3n^2 + 3n + 2$$
$$y_6 = 344.$$

3.3 Central Difference

The central difference operator denoted by δ is defined by

$$\delta y(x) = y\left(x + \frac{h}{2}\right) - y\left(x - \frac{h}{2}\right)$$

Thus $\delta y(x) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) y(x)$

Giving the operator relation, $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$ or $\delta E^{\frac{1}{2}} = E - 1$

Also $\delta y_n = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) y(x_n) = E^{\frac{1}{2}} y_n - E^{-\frac{1}{2}} y_n$

That is $\delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$

Further

$$\delta^2 y_n = \delta(\delta y_n) = \delta y_{n+\frac{1}{2}} - \delta y_{n-\frac{1}{2}}$$
$$\delta^2 y_n = (E + E^{-1} - 2) y_n$$

That is $\delta^2 = E + E^{-1} - 2$

Even though the central difference operator uses fractional arguments, still it is widely used. This is related to the averaging operator and is defined by,

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$
$$\mu^2 = \frac{1}{4}(E + 2 + E^{-1}) = \frac{1}{4}(\delta^2 + 2 + 2) = 1 + \frac{1}{4}\delta^2$$

It may be noted that,$\delta y_{\frac{1}{2}} = y_1 - y_0 = \nabla y_1$

Also, $\delta E^{\frac{1}{2}} y_1 = y_2 - y_1 = \Delta y_1$

$$\therefore \delta E^{\frac{1}{2}} = \Delta = E - 1$$

Example 1: Evaluate $\Delta \tan^{-1} x$

Solution:

$$\Delta \tan^{-1} x = \tan^{-1}(x + h) - \tan^{-1} x$$
$$\Delta \tan^{-1} x = \tan^{-1}\left\{\frac{x + h - x}{1 + (x + h)x}\right\} = \tan^{-1}\left\{\frac{h}{1 + hx + x^2}\right\}$$

Example 2: Evaluate $\Delta^2 \cos 2x$

Solution:

$\Delta^2 \cos 2x = \Delta\{\cos 2(x+h) - \cos 2x\} = \Delta \cos 2(x+h) - \Delta \cos 2x$

$= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]$

$= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h$

$= -2\sin h\ \ [2\cos(2x+2h)\sin h]$

$= -4\sin^2 h\cos(2x+2h)$

Example 3: Prove the following operator relations. (i) $\Delta = \nabla E$ (ii) $(1+\Delta)(1-\nabla) = 1$ (iii) $E = e^{hD}$

Solution:

(i) Since $\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x), \Delta = E - 1$

Or $1 + \Delta = E$ (i)

Also $\nabla f(x) = f(x) - f(x-h) = (1 - E^{-1})f(x), \nabla = 1 - E^{-1}$

Or $1 - \nabla = E^{-1}$ (ii)

Thus, $\nabla = \frac{E-1}{E}$ or $\nabla \mathrm{E} = \mathrm{E} - 1 = \Delta$

Hence $\Delta = \nabla E$

(ii) From (i) and (ii)

$$(1+\Delta)(1-\nabla) = E * E^{-1} = 1$$

(iii) $Ef(x) = f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \cdots$

$= f(x) + hDf(x) + \frac{h^2}{2!}D^2 f(x) + \cdots$

$= \left(1 + hD + \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + \cdots\right) f(x) = e^{hD}\,f(x)$

$\therefore E = 1 + \Delta = e^{hD}$

Example 4: Prove the following operator relations. (i) $\mu + \frac{1}{2}\delta = E^{1/2}$ (ii) $\mu - \frac{1}{2}\delta = E^{-1/2}$ (iii) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

Solution:

$\mu + \frac{1}{2}\delta = \frac{E^{1/2}+E^{-1/2}}{2} + \frac{E^{1/2}-E^{-1/2}}{2} = E^{1/2}$

$\mu - \frac{1}{2}\delta = \frac{E^{1/2}+E^{-1/2}}{2} - \frac{E^{1/2}-E^{-1/2}}{2} = E^{-1/2}$

$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - 1 + 1 - E^{-1}) = \frac{1}{2}(E - E^{-1}) = \mu\delta$

3.4 Detection of errors by using difference table

Suppose there is an error ε in the entry y_5 of a table. As higher differences are formed, this error spreads out and is considerably magnified. Let us see, how it effects the difference table.

- The error increases with the order of differences.
- The coefficient of $\varepsilon's$ in any column are the binomial coefficient of $(1-\varepsilon)^n$. Thus the errors in the fourth difference column are $\varepsilon, -4\varepsilon, 6\varepsilon, -4\varepsilon, \varepsilon$.
- The algebraic sum of the errors in any difference column is zero.
- The maximum error in each column, occurs opposite to the entry containing the error. That is y_5.

Example 1: One entry in the following table is incorrect and y is a cubic polynomial in x. Use the difference table to locate and correct the error.

X	0	1	2	3	4	5	6	7
Y	25	21	18	18	27	45	76	123

Solution: The difference table is given by

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	25			
		-4		
1	21		1	
		-3		2
2	18		3	
		0		6
3	18		9	
		9		0
4	27		9	
		18		4
5	45		13	
		31		3
6	76		16	
		47		
7	123			

y being a polynomial of the third degree, $\Delta^3 y$ must be constant, that is the same. The sum of the third differences being 15, each entry under $\Delta^3 y$ must be 15/5. That is 3. Thus the four entries under $\Delta^3 y$ are in error which can be written as,

$$3-1, 3-3(-1), 3+3(-1), 3-(-1)$$

Take $\varepsilon = -1$, we find that the entry corresponding to $x = 3$ is in error.

$$\therefore y + \varepsilon = 18$$

Thus the true value of y = 18 - ε = 18 – (-1) = 19.

Example 2: Using the method of separation of symbols, prove that

$$u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \cdots = e^x \left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right)$$

Solution:

$$u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \cdots = u_0 + \frac{x}{1!}\, E\, u_0 + \frac{x^2}{2!}E^2 u_0 + \frac{x^3}{3!}E^3 u_0 + \cdots$$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \cdots\right)u_0 = e^{xE}u_0$$

$$= e^{x(1+\Delta)}u_0 = e^x . e^{x\Delta}u_0$$

$$= e^x \left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \cdots\right)u_0$$

$$= e^x \left(u_0 + \frac{x}{1!}\,\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \cdots\right)$$

Lets Sum up

Finite difference methods are crucial in numerical analysis for approximating derivatives and solving differential equations. Here, we overview forward difference, backward difference, central difference, and error detection using a difference table.

❖ **Forward Difference**

Concept:

- The forward difference method approximates the derivative of a function using the difference between function values at consecutive points. It is a simple and direct method often used in initial stages of numerical differentiation.

Advantages:

- Easy to implement.
- Requires fewer function evaluations compared to other methods.

Disadvantages:

- Less accurate, especially for functions with high curvature.
- Errors can propagate and amplify in successive calculations.

❖ **Backward Difference**

Concept:

- The backward difference method uses the difference between the current and previous function values to approximate the derivative. It is particularly useful when data points are known from the past.

Advantages:

- Similar simplicity to the forward difference.
- Can be used effectively in backward-time stepping problems.

Disadvantages:

- Like forward difference, it is less accurate for highly curved functions.
- May not handle initial boundary conditions well.

❖ **Central Difference**

Concept:

- The central difference method averages the forward and backward differences to approximate the derivative. This method provides a more accurate estimate by considering the symmetric difference around the point of interest.

Advantages:

- Higher accuracy compared to forward and backward differences.
- Better error properties, especially for smooth functions.

Disadvantages:

- Requires more function evaluations.
- May not be applicable at the boundaries of the data set.

❖ **Error Detection Using Difference Table**

Concept:

- A difference table helps detect errors in numerical differentiation and interpolation. By examining the higher-order differences, one can identify inconsistencies and potential errors in the data or the numerical method.

Advantages:

- Provides a systematic way to check for errors.
- Helps in identifying trends and patterns in data.

Disadvantages:

- Construction of the table can be time-consuming for large data sets.
- Requires careful interpretation to diagnose errors accurately.

Conclusion

Finite difference methods, including forward, backward, and central differences, are essential tools in numerical analysis for approximating derivatives. Each method has its specific advantages and limitations, making them suitable for different types of problems. Error detection using a difference table adds a layer of robustness to numerical calculations, ensuring greater accuracy and reliability in results.

Understanding and applying these methods effectively is key to solving a wide range of numerical problems in science and engineering.

Self Assessment Questions:

1. Find the sixth term of the sequence 8, 12, 19, 29, 42, …
2. Find f(x) from the table below. Also find f(7).

X	0	1	2	3	4	5	6
f(x)	-1	3	19	53	111	199	323

3. The following table gives the value of y which is a polynomial of degree five. It is known that $f(3)$ is in error. Correct the error.

X	0	1	2	3	4	5	6
Y	1	2	33	254	1025	3126	7777

4. Using the method of separation of symbols, prove that $u_1x + u_2x^2 + u_3x^3 + \cdots = \frac{x}{1-x}u_1 + \left(\frac{x}{1-x}\right)^2 \Delta u_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 u_1 + \cdots$

Answers for check-up your progress:

1. 58, 2. f(x) = x^3+3x^2-1 3. $\Delta^2 f(x) = 3^7(8)(7)\left(x+\frac{19}{3}\right)^{(6)} + 2(3^6)(7)(6)\left(x+\frac{19}{3}\right)^{(5)}$
4. 10

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2. B. S. Grewal, Numerical Methods in Engineering and Science with Programs in C, C++ and MATLAB, Khanna Publishers.
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Unit – 4

INTERPOLATIONS

4.0 Introduction

Interpolation is a method of deriving a simple function from the given discrete data set such that the function passes through the provided data points. This helps to determine the data points in between the given data ones. This method is always needed to compute the value of a function for an intermediate value of the independent function. In short, interpolation is a process of determining the unknown values that lie in between the known data points. It is mostly used to predict the unknown values for any geographical related data points such as noise level, rainfall, elevation, and so on.

4.1 Differences of Polynomial

The nth differences of a polynomial of the nth degree are constant and all higher order differences are zero. Let the polynomial of the nth degree in x, be

$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l$

$$\therefore \Delta f(x) = f(x+h) - f(x)$$

$$= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \cdots + kh$$

$$= anhx^{n-1} + b' x^{n-2} + c' x^{n-3} + \cdots + k' x + l'$$

Where $b', c', \ldots, l'$ are the new constant coefficients. Thus the first differences of a polynomial of the nth degree is a polynomial of degree (n-1).

Similarly,

$\therefore \Delta^2 f(x) = \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x)$

$= anh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \cdots + k'h$

$= an(n-1)h^2 x^{n-2} + b'' x^{n-3} + c'' x^{n-4} + \cdots + k''$,

Therefore the second differences represent a polynomial of degree $(n-2)$.

Continuing this process, for the nth differences we get a polynomial of degree zero. That is

$$\Delta^n f(x) = an(n-1)(n-2) \ldots 1.h^n = ah^n n!$$

Which is a constant. Hence the (n+1)th and higher differences of a polynomial of nth degree will be zero.

Example 1: Evaluate $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$

Solution:

$$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = \Delta^{10}[abcdx^{10} + (\quad)x^9 + (\quad)x^8 + \cdots + 1]$$

$$= abcd\ \Delta^{10}(x^{10}) = abcd\ (10\,!)$$

Example 2: Evaluate $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$ if h = 2.

Solution:

$$\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)] = \Delta^{10}[24x^{10} + terms\ of\ lesser\ degree]$$

$$= 24(10!)2^{10} + 0$$

$$= 24(10!)2^{10}$$

Example 3: Find $\Delta^3 f(x)$ if $f(x) = (3x+1)(3x+4)(3x+7)\ldots(3x+19)$

Solution:

Given $f(x) = (3x+1)(3x+4)(3x+7)\ldots(3x+19)$ contains 7 factors

$= 3^7\left(x+\frac{1}{3}\right)\left(x+\frac{4}{3}\right)\ldots\left(x+\frac{19}{3}\right)$

$= 3^7\left(x+\frac{19}{3}\right)^{(7)}$

$\Delta f(x) = 3^7\,(7)\left(x+\frac{19}{3}\right)^{(6)}$

$\Delta^2 f(x) = 3^7\,(7)(6)\left(x+\frac{19}{3}\right)^{(5)}$

$\Delta^3 f(x) = 3^7\,(7)(6)(5)\left(x+\frac{19}{3}\right)^{(4)}$

4.2 Interpolation

Many times, data is given only at discrete points such as (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n). So, how then does one find the value of y at any other value of x Well, a continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points (Figure 1). Then one can find the value of y at any other value of x. This is called *interpolation*. Of course, if x falls outside the range of x for which the data is given, it is no longer interpolation but instead is called *extrapolation*.

So what kind of function $f(x)$ should one choose. A polynomial is a common choice for an interpolating function because polynomials are easy to

(A) evaluate,

(B) differentiate, and

(C) integrate,

relative to other choices such as a trigonometric and exponential series.

4.3 Newton forward and Newton Backward differences:

The Newton's forward interpolation formula is

$$y(x_0+nh)=y_0+n\Delta y_0+\frac{n(n-1)}{2!}\Delta^2 y_0+\frac{n(n-1)(n-2)}{3!}\Delta^3 y_0+.....$$

The Newton's backward interpolation formula is

$$y(x_0+nh)=y_0+n\nabla y_0+\frac{n(n+1)}{2!}\nabla^2 y_0+.....$$

Example 1: From the data given below, find the number of students whose weight is between 60 to 70

Weight	0-40	40-60	60-80	80-100	100-120
No of Students	250	120	100	70	50

Solution:

x Weight	Y No of Students	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		-20		
		100		-10	
Below 80	470		-30		20
		70		10	
Below 100	540		-20		
		50			
Below 120	590				

$$u=\frac{x-x_0}{h}=\frac{70-40}{20}=1.5$$

$$y(70)=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+.......$$

$$=250+(1.5)(120)+\frac{(1.5)(1.5)}{2}(-20)+\frac{(0.5)(0.5)(0.5)}{6}(-10)$$

$$+\frac{(1.5)(0.5)(-0.5)(-1.5)}{24}$$

$$=424$$

∴ Number of students whose weight is between 60 and 70 = y(70) – y(60) = 424 – 370 = 54

Example 2: A function f(x) is given by the following table. Find f(0.2) by a suitable formula**.**

x	0	1	2	3	4	5	6
F(x)	176	185	194	203	212	220	229

Solution:

The difference table is follows:-

x	y = f(x)	$\Delta^1 y_0$	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$	$\Delta^6 y_0$
0	176						
1	185	9					
2	194	9	0				
3	203	9	0	0			
4	212	9	0	0	0		
5	220	8	-1	-1	-1	-1	
6	229	9	1	2	3	4	5

Here $x_0 = 0$, h = 1, $y_0 = 176 = f(x)$

We have to find the value of f (0.2). By Newton's forward interpolation formula we have

$$f(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + ...$$

$$f(0.2) = ?$$

$x_0 + nh = 0.2$

$0 + n = 0.2 \Rightarrow \quad n = 0.2$

$$f(0.2) = 176 + (0.2)\ 9 + \frac{(0.2)(0.2-1)}{2}\ 0$$

$$= 176 + 1.8$$

$$= 177.8$$

Example 3. From the given table compute the value of sin 38.

x	0	10	20	30	40
Sin x	0	0.17365	0.34202	0.5	0.64276

Solution:

As we have to determine the value of y = sin x near the lower end, we apply Newton's backward interpolation formula.

The difference table is as given below.

x_0	y(x) = Sin x_0	Δ_y	Δ^2_y	Δ^3_y	Δ^4_y
0	0				
10	0.17365	0.17365			
20	0.34202	0.16837	-0.00528		
30	0.5000	0.15798	-0.01039	-0.00511	
40	0.64279	0.14279	-0.01519	-0.0048	0.00031
X_0	Y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$

Here x_0 = 40, h = 0.64279, h = 10

Newton's backward differences formula

$$y(x_0 + nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + ...$$

$$y(38) =$$

$$x_0 + nh = 38 \qquad 40 + n(10) = 38 \qquad \boxed{n = -0.2}$$

$$y\ (38) = 0.64279 + (-0.2)\ (0.14279) + \frac{(-0.2)\ (-0.2+1)}{2!}\ (-0.01519)$$

$$+\frac{(-0.2)\ (-0.2+1)\ (-0.2+2)}{3!}\ (-0.0048) + \ neglible\ term$$

$$= 0.64279 - 0.028558 + 0.0012152 + 0.0002304$$

$$= 0.61566$$

Example 4. In an examination the number of candidates who obtained marks between certain limits were as follows:-

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of Students	31	42	51	35	31

Find the number at candidate whose scores lie between 45 and 50.

Solution:

First we construct a cumulative frequency table for the given table.

Upper limits	*40*	*50*	*60*	*70*	*80*
C.F.	31	73	124	159	190

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
50	73	42			
60	124	51	9		
70	159	35	-16	- 25	
80	190	31	- 4	12	37

We have $x_0 = 40$, $x = 45$, $h = 10$

$$U = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

$y_0 = 73$, $\Delta y_0 = 42$, $\Delta^2 y_0 = 9$, $\Delta^3 y_0 = - 25$, $\Delta^4 y_0 = 37$

From Newton's forward interpolation formula

$$f(x) = y_0 + U\Delta y_0 + \frac{U(U-1)}{2!}\Delta^2 y_0 + \frac{U(U-1)\,(U-2)}{3!}\Delta^3 y_0 + \frac{U(U-1)\,(U-2)\,(U-3)}{4!}\Delta^4 y_0 + \ldots\ldots$$

$$f(45) = 31 + (0.5)\ 42 + \frac{(0.5)\,(-0.5)}{2!}\,(9) + \frac{(0.5)\,(0.5-1)\,(0.5-2)}{6}\,(-25) + \frac{(0.5)\,(-0.5)\,(-1.5)\,(-2.5)}{24}\,(37)$$

$= 47.8673 = 48$ approximately.

The number of students who obtained marks less than 45 = 48, and the number of students who scored marks between 45 and 50 = 73 – 48 = 25.

Example 5: Use Newton's forward interpolation and find value of sin 52 from the following data.

X	45	50	55	60
Y=sinx	0.7071	0.7660	0.8192	0.8660

Solution:

The difference table

X	Sin x	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0462		
60	0.8660			

We have $x_0 = 45$, $x_1 = 52$, $y_0 = 0.7071$, $\Delta y_0 = 0.0589$, $\Delta^2 y_0 = -0.0057$, $\Delta^3 y_0 = -0.0007$

$$u = \frac{x - x_0}{h} = \frac{52° - 45°}{5°} = 1.4$$

Newton's formula

$$y = u_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

$$f(52) = 0.7071 + 1.4(0.0589) + \frac{(1.4)(1.4-1)}{2}(-0.0057)$$

$$+\frac{(1.4)(1.4-1)(1.4-2)}{3!}(-0.0007)$$

=0.7071+0.8246-0.001596+0.000392

$$\sin 52° = 0.7880032$$

Example 6: Write the polynomial to calculate the value of x when?

X	3	5	7	9
Y	6	24	58	108

Solution:

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
3	6			
		18		
5	24		16	
		34		0
3	58		16	
		50		
9	108			

$$x_0 + nx = x$$

$$3 + n2 = x; \quad 2n = x - 3; \quad n = \frac{x-3}{2}$$

$$y(x_0+nx)=y_0+n\Delta y_0+\frac{n(n-1)}{2!}\Delta^2 y_0+.....$$

$$y(x)=6+\left(\frac{x-3}{2}\right)18+\frac{\left(\frac{x-3}{2}\right)\left(\frac{x-3}{2}-1\right)}{2}(16)$$

$$y(x)=2x^2-3x+9$$

4.4 Central Difference Interpolation formula

In the preceding sections, we derived Newton's forward and backward interpolation formula which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formula which are best suited for interpolation near the middle of the table.

4.4.1 Gauss Forward interpolation Formula

By using Newton's forward interpolation formula, we can derive the following Gauss forward interpolation formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots$$

4.4.2 Gauss Backward interpolation Formula

By using Newton's backward interpolation formula, we can derive the following Gauss backward interpolation formula,

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots$$

4.4.3 Stirling's Formula

The mean of Gauss forward interpolation formula and Gauss backward interpolation formula is

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} * \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

4.4.4 Bessel's Formula

From the Gauss forward interpolation formula, we can derive the Bessel's formula is given by

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p-\frac{1}{2}\right)p\,(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p+2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\dots$$

4.4.5 Everett's Formula

From the Gauss forward interpolation formula, we can derive the Everett's formula is given by

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots$$

Where p = 1 – q

There is a close relationship between Bessel's formula and Everett's formula and one can be deduced from the other by suitable rearrangements. It is also interesting to observe that Bessel's formula truncated after third differences is Everett's formula truncated after second differences.

Example 1: Find $f(22)$ from the Gauss forward formula

X	20	25	30	35	40	45
F(x)	354	332	291	260	231	204

Solution:

Taking $x_0 = 25, h = 5$, we have to find the value of f(x) for x = 22.

That is for $p = \frac{x-x_0}{h} = \frac{22-25}{5} = -0.6$

The difference table is a follows

x	p	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
20	-1	354					
			-22				
25	0	332		-19			
			-41		29		
30	1	291		10		-37	
			-31		-8		45
35	2	260		2		8	
			-29		0		
40	3	231		2			
			-27				
45	4	204					

Gauss forward formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots$$

$\therefore f(22) = 332 + (0.6)(-41) + \frac{(-0.6)(-0.6-1)}{2!}(-19) + \cdots$

$= 332 + 24.6 - 9.12 - 0.512 + 1.5392 - 0.5241$

Hence $f(22) = 347.983$

Example 2: Using Gauss backward difference formula, find $f(8)$ from the following table

X	0	5	10	15	20	25
f(x)	7	11	14	18	24	32

Solution:

Taking $x_0 = 10, h = 5$, we have to find the value of y for x = 8.

That is for $p = \frac{x-x_0}{h} = \frac{8-10}{5} = -0.4$

The difference table is a follows

x	p	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-2	7				
			4			
5	-1	11		-1		
			3		2	
10	0	14		1		-1
			4		1	
15	1	18		2		-1
			6		0	
20	2	24		2		
			8			
25	3	32				

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \cdots$$

$f(8) = 14 + (-0.4)(3) + \frac{(-0.4+1)(-0.4)}{2!}(1) + \cdots = 14 - 1.2 - 0.12 + 0.112 + 0.034$

Hence $f(8) = 12.826$

Example 3: Given

θ	0	5	10	15	20	25	30
$\tan\theta$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of $\tan 16°$.

Solution:

Taking the origin at $\theta° = 15°, h = 5°$ and $p = \frac{\theta - 15}{5}$

We have the following central difference table:

p	$f(x) = \tan\theta$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-3	0				
		0.0875			
-2	0.0875		0.013		
		0.0888		0.0015	
-1	0.1763		0.0028		0.0002
		0.0916		0.0017	
0	0.2679		0.0045		0.0000
		0.0961		0.0017	
1	0.3640		0.0062		0.0009
		0.1023		0.0026	
2	0.4663		0.0088		
		0.1111			
3	0.5774				

At $\theta° = 16°, h = 5°$ and $p = \frac{16-15}{5} = 0.2$

By stirling's formula

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} * \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$y_{0.2} = 0.2679 + (0.2)\left(\frac{0.0916+0.0961}{2}\right) + \frac{(0.2)^2}{2!}\ (0.0045) + \cdots$

$= 0.2679 + 0.01877 + 0.00009 + \cdots$

Hence $\tan 16° = 0.28676.$

Example 4: Apply Bessel's formula to find the value of $f(27.5)$ from the table

X	25	26	27	28	29	30
F(x)	4	3.846	3.704	3.571	3.448	3.333

Solution:

Taking the origin at $x_0 = 27, h = 1$, we have $p = x - 27$

The central difference table is

x	p	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
25	-2	4.000				
			- 0.154			
26	-1	3.846		0.012		
			- 0.142		-0.003	
27	0	3.704		0.009		0.004
			-0.133		-0.001	
28	1	3.571		0.010		-0.001
			-0.123		-0.002	
29	2	3.448		0.008		
			-0.115			
30	3	3.333				

At $x = 27.5, p = 0.5$ (As p lies between ¼ and ¾)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p-\frac{1}{2}\right)p\ (p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p+2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\ldots$$

Since p = 0.5, we have

$y_p = 3.704 - \frac{(0.5)(0.5-1)}{2}\left(\frac{0.009+0.010}{2}\right) + 0 + \cdots$

$= 3.704 - 0.11875 - 0.00006$

Hence $f(27.5) = 3.585$

Example 5: Using Everett's formula, evaluate $f(30)\ if\ f(20) = 2854, f(28) = 3162,$ $f(36) = 7088,\ f(44) = 7984$.

Solution:

Taking the origin at $x_0 = 28, h = 8$, we have $p = \frac{x-28}{8}$

The central difference table is

x	p	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
20	-1	2854			
			308		
28	0	3162		3618	
			3926		-6648
36	1	7088		-3030	
			896		
44	2	7984			

At $x = 30, p = \frac{30-28}{8} = 0.25$ and $q = 1 - p = 0.75$

Everett's formula is

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \cdots + py_1$$
$$+ \frac{p(p^2-1^2)}{3!}\Delta^2 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \cdots$$

$= (0.75)(3162) + \frac{(0.75)(0.75^2-1)}{6}(3618) + \cdots + (0.25)(7080) + \frac{(0.25)(0.25^2-1)}{6}(-3030) + \cdots$

$= 2371.5 - 351.75 + 1770 + 94.69$

Hence $f(30) = 3884.4$

Lets Sum Up

- **Newton's Forward Difference Formula**

Concept:

Used for polynomial interpolation, this method calculates the interpolated value at a point by utilizing the forward differences of function values at equally spaced points.

Advantages:

- Simplicity: Easy to understand and implement.
- Good for Smooth Functions: Works well with smooth and continuous functions.
- Efficient for Interpolation: Effective when predicting values near the start of the dataset.

Disadvantages:

- Oscillation: The polynomial can oscillate for larger intervals, leading to inaccuracies.
- Recalculation Needed: Forward differences must be recalculated for new data points.

❖ **Newton's Backward Difference Formula**

Concept:

Similar to the forward method, but it uses backward differences. This approach estimates values using known values at points before the desired point.

Advantages:

- Extrapolation: More accurate when estimating points close to the last data point.
- Equally Spaced Data: Works well with uniformly spaced data points.

Disadvantages:

- Limited to Endpoints: Less efficient for interpolation at the beginning of the dataset.
- Oscillation: Same oscillation issue as the forward method with larger intervals.

❖ **Gauss Central Difference Formula**

Concept:

This method employs central differences, taking points both before and after the target to estimate values, providing a more balanced interpolation.

Advantages:

- Higher Accuracy: Generally yields better results than forward or backward methods for centrally located points.
- Stable: Reduces error in polynomial approximation.

Disadvantages:

- Symmetrical Data Required: Requires data points to be evenly spaced around the interpolation point.

- Complex Calculations: More involved calculations compared to forward/backward methods.

❖ **Stirling's Formula**

Concept:

A specific case of the central difference method tailored for evenly spaced data, particularly effective for interpolation at midpoints.

Advantages:

- Balanced Approach: Good for midpoints in datasets.
- Efficiency: Less computationally intensive compared to higher-degree polynomial interpolations.

Disadvantages:

- Odd Number of Points: Requires an odd number of data points for optimal performance.
- Not Suitable for Unequal Spacing: Less effective with unevenly spaced data.

❖ **Everett's Formula**

Concept:

This formula is designed for interpolation when data points are not uniformly spaced, focusing on local approximations.

Advantages:

- Flexibility: Can handle non-uniformly spaced datasets effectively.
- Local Approximation: Provides localized interpolation based on surrounding data points.

Disadvantages:

- Complexity: More complex to derive and implement.
- Potential Accuracy Issues: May be less accurate than other methods for well-behaved datasets.

❖ **Bessel's Formula**

Concept:

Bessel’s formula is used for interpolation and extrapolation, particularly effective for equally spaced datasets.

Advantages:

- High Accuracy: Performs well with periodic and well-behaved functions.
- Good for Interpolation: Suitable for estimating values not present in the dataset.

Disadvantages:

- Uniform Spacing Required: Less effective for non-uniform data distributions.
- Computational Resources: Can be more resource-intensive due to complexity.

Conclusion

In conclusion, each interpolation method has its unique strengths and weaknesses, making them suitable for different scenarios:

- Newton's Forward and Backward methods are ideal for simple, evenly spaced data, with forward suitable for starting estimates and backward for endpoints.
- Gauss Central offers better accuracy, especially for central points, but requires symmetric spacing.
- Stirling's Formula is efficient for midpoint interpolation with evenly spaced data.
- Everett's Formula provides flexibility for unevenly spaced datasets but at the cost of complexity.
- Bessel's Formula excels with periodic functions but requires uniform data.

Choosing the right method depends on the nature of your data, the desired accuracy, and the specific interpolation requirements. For smooth functions with evenly spaced points, Newton's methods and Stirling's are effective; for central estimates, Gauss and Bessel shine; while Everett's is beneficial for irregular datasets.

Self Assessment Questions:

1. Evaluate: $\Delta^{10}[(1-x)(1-2x)(1-3x)\ldots(1-10x)]$ if h = 1.
2. Find $\Delta^2 f(x)$ if $f(x) = x(3x+1)(3x+4)(3x+7)\ldots(3x+19)$
3. Find the values of y at x=21 and x=28 from the following data by using Newton's forward and backward formula

X	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

4. Use Gauss's forward formula to evaluate y_{30}, given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070, y_{33} = 16.3432$ *and* $y_{37} = 15.5154$

5. Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that

Year	1939	1949	1959	1969	1979	1989
Population (in thousands)	12	15	20	27	39	52

6. Employ Stirling's formula to compute $y_{12.2}$ from the following table

X	10	11	12	13	14
Y	23967	28060	31788	35209	38368

7. Apply Bessel's formula to obtain y_{25} given $y_{20} = 2854, y_{24} = 3162, y_{28} = 3544,$ $y_{32} = 3992$.
8. Given the table

X	310	320	330	340	350	360
log x	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

Find the value of $\log\ 337.5$ by Everett's formula.

Answers:

1. $(10!)^2$ 2. $\Delta^2 f(x) = 3^7(8)(7)\left(x+\frac{19}{3}\right)^{(6)} + 2(3^6)(7)(6)\left(x+\frac{19}{3}\right)^{(5)}$, 3. 0.3583 & 0.4695, 4. 16.9216 5. 32.532, 6. 0.32497, 7. 3250.875, 8. 2.5283

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Unit 5

INTERPOLATION AND APPROXIMATIONS

5.0 Introduction

Interpolation is a method of fitting the data points to represent the value of a function. It has a various number of applications in engineering and science, that are used to construct new data points within the range of a discrete data set of known data points or can be used for determining a formula of the function that will pass from the given set of points (x,y).

5.1 Lagrange's Interpolation

Polynomial interpolation involves finding a polynomial of order n that passes through the $n+1$ data points. One of the methods used to find this polynomial is called the Lagrangian method of interpolation. Other methods include Newton's divided difference polynomial method and the direct method.

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $n-1$ terms with terms of $j = i$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example 1:

The upward velocity of a rocket is given as a function of time in Table 1.

Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the value of the velocity at $t = 16$ seconds using a first order Lagrange polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), the velocity is given by

$$v(t) = \sum_{i=0}^{1} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1)$$

Since we want to find the velocity at $t = 16$, and we are using a first order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

Then

$$t_0 = 15, \; v(t_0) = 362.78$$

$$t_1 = 20, \; v(t_1) = 517.35$$

gives

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{t - t_j}{t_0 - t_j}$$

$$= \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{t - t_j}{t_1 - t_j}$$

$$= \frac{t - t_0}{t_1 - t_0}$$

Hence

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1)$$

$$= \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35), \quad 15 \le t \le 20$$

$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35)$$

$$= 0.8(362.78) + 0.2(517.35)$$

$$= 393.69 \text{ m/s}$$

You can see that $L_0(t) = 0.8$ and $L_1(t) = 0.2$ are like weightages given to the velocities at $t = 15$ and $t = 20$ to calculate the velocity at $t = 16$.

5.2 Formula for Lagrange's interpolation.

Let Y = f(x) be a function which assumes the values $f(x_0)$, $f(x_1)$ ….. $f(x_n)$ corresponding to the values x: x_1, x_1 …..x_n.

$$Y = f(x) = \frac{(x - x_1)(x - x_2)...(x - x_n)}{(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)...(x - x_n)}{(x_1 - x_0)(x_1 - x_2)...(x_1 - x_n)} f(x_1) + ...$$

Example 1: Find the second degree polynomial fitting the following data.

x	**1**	**2**	**4**
y	**4**	**5**	**13**

Solution:-

$x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $y_0 = 4$, $y_1 = 5$, $y_2 = 13$

By Lagrange's formula

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_0)} y_2$$

$$= \frac{(x-2)(x-4)}{3} 4 + \frac{(x-1)(x-4)}{-2} 5 + \frac{(x-1)(x-2)}{6} 13$$

$$= \frac{1}{6}\left[8x^2 - 48x + 64 - 15x^2 + 75x - 60 + 13x^2 - 39x + 26\right]$$

$$= \frac{1}{6}\left[6x^2 - 12x + 30\right]$$
$$= x^2 - 2x + 5$$

Example 2: Using Lagrange's interpolation formula, find the value of y corresponding to x = 10 from the following table.

x	5	6	9	11
f(x)	12	13	14	16

Solution:-

We have $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$

$Y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$

Using Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)\,(x-x_2)\,(x-x_3)}{(x_0-x_1)\,(x_0-x_2)\,(x_0-x_3)} y_0 + \frac{(x-x_0)\,(x-x_2)\,(x-x_3)}{(x_1-x_0)\,(x_1-x_2)\,(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Substitute

$$f(10) = \frac{(10-6)\,(10-9)\,(10-11)}{(5-6)\,(5-9)\,(5-11)}(12) + \frac{(10-5)\,(10-9)\,(10-11)}{(6-5)\,(6-9)\,(6-11)}(13)$$
$$+ \frac{(10-5)\,(10-6)\,(10-11)}{(9-6)\,(9-6)\,(9-11)} 14 + \frac{(10-5)\,(10-6)\,(10-9)}{(11-5)\,(11-6)\,(11-9)} \times 16$$
$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{42}{3}$$

Example 3: Show that $\underset{bcd}{\overset{3}{\Delta}}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

Solution:

If f(x) = 1/x, f(a) = 1/a

f(a, b) = $\underset{b}{\Delta}\left(\frac{1}{a}\right) = \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = -\frac{1}{ab}$

f(a, b, c) $= \frac{f(b,c) - f(a,b)}{c-a} = \frac{-\frac{1}{bc} + \frac{1}{ab}}{c-a}$

$$= \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{\frac{1}{\text{bcd}} - \frac{1}{abc}}{\text{d-a}} = -\frac{1}{abcd}$$

5.3 Newton's Divided Differences

To illustrate this method, we will start with linear and quadratic interpolation, then, the general form of the Newton's Divided Difference Polynomial method will be presented.

Given (x_0, y_0), (x_1, y_1), fit a linear interpolant through the data. Note taht $y_0 = f(x_0)$ and $y_1 = f(x_1)$, assuming a linear interpolant means:

The first divided difference by ,

$$f[x_0] = f(x_0)$$

The second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

And the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets. Then, We can write:

$$f(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + \ldots$$

Example 1: Find the form of the function f(x) under suitable assumption from the following

X	0	1	2	5
f(x)	2	3	12	147

Solution:

The divided differences table is given below.

x	**f(x)**	**Δf(x)**	**Δ^2f(x)**	**Δ^3f(x)**
0	2			
1	3	$\frac{3-2}{1-0}=1$		
2	12	$\frac{12-3}{2-1}=9$	$\frac{9-1}{2-0}=4$	
5	147	$\frac{147-12}{5-2}=45$	$\frac{45-9}{5-1}=9$	$\frac{9-4}{5-0}=1$

We have $x_0 = 0$, f(x_0 = 2, f(x_0, x_1) = 1, f(x_0, x_1, x_2) = 4 , f(x_0, x_1, x_2, x_3) = 1

The Newton's divided difference interpolation formula is

$$f(x)=(fx_0)+(x-x_0)f(x_0,x_1)+(x-x_0)\ (x-x_1)f(x_0,x_1,x_2)$$
$$+\ (x-x_0)\ (x-x_1)\ (x-x_2)f(x_0,x_1,x_2,x_3)$$
$$=2+(x-0)\ 1+(x-0)\ (x-1)\ 4+(x-0)\ (x-1)\ (x-2)\ 1$$
$$=x^3+x^2-x+2$$

Example 2: Find f(x) as a polynomial in x for the following data by Newton's divided difference formula

X :	-4	-1	0	2	5
f(x) :	1245	33	5	9	1335

Solution:

The divided difference table is

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		- 404			
-1	33		94		
		- 28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

By Newton's divided difference interpolation formula

f(x)= 1245 + (x+4) (-404) + (x+4) (x+1) 94 + (x+4)

(x+1) x(-14) + (x+4) (x+1) x(x-2)3 = $3x^4 + x^3 - 14x+5$

Example 3:

The following table gives same relation between steam pressure and temperature. find the pressure at temperature 372.1^0

T	361^0	367^0	378^0	387^0	399^0
P	154.9	167.9	191.0	212.5	244.2

Solution:

The divided difference table is

T	P	Δp	$\Delta^2 p$	$\Delta^3 P$	$\Delta^4 p$
361	154.9				
		2.016666			
367	167.0		0.0097147		
		2.18181818		0.000024	
378	191.0		0.0103535		0.00000073
		2.388889		0.000052	
387	212.5		0.01203703		
		2.641667			
399	244.2				

By Newton's divided difference formula

$P(T=372.1^0)$ = 154.9 + (11.1)(2.016666) + (11.1)

(5.1) (0.009914) + (11.1) (5.1) (-5.9) (0.000024)

+ (11.1) (5.1) (-5.9) (-14.9) (0.00000073) = 177.8394819

5.4 Inverse Lagrange's interpolation:

By using Lagrange's method we can find the value of x when f(x) is given. The inverse Lagrange's formula is given by,

$$x = \frac{(y-y_1)(y-y_2)...(y-y_n)}{(y_0-y_1)(y_0-y_2)...(y_0-y_n)} x_0 +$$
$$\frac{(y-y_0)(y-y_2)...(y-y_n)}{(y_1-y_0)(y_1-y_2)...(y_1-y_n)} x_1 + ...$$

Example 1: Find the value of x when y = 85, using Lagrange's formula from the following table.

X	2	5	8	14
Y	94.8	87.9	81.3	68.7

Solution:-

$x_0 = 2, x_1 = 5, x_2 = 8, x_3 = 14$

$y_0 = 94.8, y_1 = 87.9, y_2 = 81.3, y_3 = 68.7$

$\because$ y = 85

We know that the Lagrange's inverse formula is

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

Substituting the above values we get,

$$x = \frac{(85-87.9)(85-81.3)(85-68.7)}{(94.8-87.9)(94.8-81.3)(94.8-68.7)}\times 2$$

$$+\frac{(85-94.8)(85-81.3)(85-68.7)}{(87.9-94.8)(87.9-81.3)(87.9-68.7)}.5$$

$$+\frac{(85-94.8)(85-87.9)(85-68.7)}{(81.3-94.8)(81.3-87.9)(81.3-68.7)}(8)$$

$$+\frac{(85-94.8)(85-87.9)(85-68.7)}{(68.7-94.8)(68.7-87.9)(68.7-81.3)}14$$

x = 6.5928.

Lets Sum Up

❖ **Lagrange's Interpolation Formula**

Concept:

Lagrange's interpolation is a polynomial interpolation method that constructs a polynomial that passes through a given set of data points. The formula is based on the concept of constructing basis polynomials for each data point.

Advantages:

- Direct Method: Provides an explicit formula for interpolation without needing to calculate divided differences.

- Easy to Understand: Intuitive and straightforward conceptually.
- Global Approximation: The polynomial constructed passes exactly through all given points.

Disadvantages:

- Computationally Intensive: Requires O(n2)O(n^2)O(n2) operations, which can be inefficient for large datasets.
- Numerical Instability: Large values of nnn can lead to Runge's phenomenon, where oscillations occur between points.
- Higher Degree Polynomial: The resulting polynomial can be of degree nnn, which may not be necessary for interpolation.

❖ **Newton's Divided Difference Formula**

Concept:

Newton's divided difference formula builds an interpolating polynomial incrementally using divided differences, which are calculated based on the data points.

Advantages:

- Incremental Construction: New points can be added easily without recalculating the entire polynomial.
- Stable for Larger Datasets: Generally more stable than Lagrange for larger datasets.
- Lower Degree Polynomials: Can achieve the same accuracy with lower degree polynomials.

Disadvantages:

- Complexity in Calculation: Requires understanding of divided differences, which can be more complex to grasp.
- Dependence on Order of Points: The order in which points are arranged can affect the results.

❖ **Inverse Lagrange's Interpolation Formula**

Concept:

This formula is used when you want to find the value of an independent variable xxx corresponding to a given dependent variable yyy. It essentially reverses the traditional Lagrange interpolation.

Advantages:

- Useful for Mapping: Allows for the direct calculation of the independent variable given the dependent variable.

- Similar Benefits to Lagrange: Retains the benefits of Lagrange interpolation, including exactness at data points.

Disadvantages:

- Computational Complexity: Still involves calculating basis polynomials, which can be computationally intensive.
- Numerical Instability: Similar to Lagrange, can suffer from instability and oscillation issues.

Conclusion

In summary, each interpolation method serves different purposes and has unique characteristics:

- Lagrange's Interpolation is straightforward and intuitive, making it suitable for small datasets, but can become computationally intensive and unstable for larger sets.
- Newton's Divided Difference Formula is efficient for larger datasets, allowing for easy updates and typically more stable results. It requires a more complex understanding of divided differences.
- Inverse Lagrange's Interpolation is valuable for scenarios where you need to find the independent variable given a dependent value, retaining the strengths of Lagrange while introducing additional complexity.

When selecting an interpolation method, consider the size of your dataset, the nature of your data (equally or unequally spaced), and whether you require direct or inverse interpolation. Each method has its best-fit applications based on these factors.

Self Assessment Questions:

1. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x	0	1	2	5
f(x)	2	3	12	147

2. Find the missing term in the following table using interpolation

x	0	1	2	3	4
y	1	3	9	---	81

3. Given the values

X	5	7	11	13	17
F(x)	150	392	1452	2366	5202

Evaluate f(9), using Newton's divided difference formula.

4. The following table gives the values of x and y

X	1.2	2.1	2.8	4.1	4.9	6.2
Y	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to y = 12, using inverse Lagrange's method.

Answers

1. 35, 2. 31, 3. 810, 4. 3.55

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E-Resources

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